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New Approach to the Correlation Spectrum near Intermittency: A Quantum Mechanical Analogy

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The correlation spectrum of fully developed one-dimensional mappings is studied near and at a weakly intermittent situation. Using a suitable infinite-matrix representation, the eigenvalue equation of the Frobenius–Perron operator is approximately reduced to the radial Schrödinger equation of the hydrogen atom. Corrections are calculated by quantum mechanical perturbation theory. Analytical expressions for the spectral properties and correlation functions are derived and checked numerically. Compared to our previous work, the accuracy of the present results is significantly higher owing to the controlled and systematic approximation scheme.

KEY WORDS: Chaos; intermittency; correlation spectrum; Frobenius–Perron operator; quantum mechanics; s states in the hydrogen atom.

1. INTRODUCTION

Typical intermittent behavior involves an irregular switching between regular and chaotic motion. As a result, correlation decay in intermittent systems is intermediate between that of regular and "purely" chaotic systems: it follows typically a power law.[1] Hereafter we consider intermittency in case of one-dimensional noninvertible mappings. Correlation functions may then be expressed in terms of the spectral properties of the Frobenius–Perron operator $\hat{H}$, defined by

$$(\hat{H}\varphi)(x) = \int dy\, \delta(x - f(y))\, \varphi(y) = \sum_a \frac{\varphi(f_a^{-1}(x))}{|f'(f_a^{-1}(x))|} \tag{1}$$

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Here $\varphi(x)$ is an arbitrary function, $f(x)$ stands for the mapping function and a labels the inverse branches. The power law decay is associated with an accumulation point at the upper edge of the spectrum that causes serious difficulties at direct numerical studies of the spectral properties. Moreover, we have found that the corresponding eigenvectors tend to become approximately parallel, that greatly reduces the number of eigenstates which can be extracted from a finite size matrix representation.[2] Therefore, a suitable analytical treatment is needed which does not involve any truncations. Such a method has already been applied in refs. 2 and 3. Here we present another approach which is based on a suitable infinite matrix representation. This method has the advantage that it approximately reduces the problem to the well known radial Schrödinger equation for s-states. Mathematically significant is the fact that the original problem is transformed into an equivalent one which admits a nearly hermitian representation. Therefore, beyond the interesting analogy, our method renders possible the application of standard perturbation theory as a tool for a systematic calculation of corrections. This results in a rather high accuracy for the spectral properties, as we shall demonstrate.

We develop our method for the treatment of the family of piecewise parabolic maps[4]

$$x' = \frac{1}{2r}\left(1 + r - \sqrt{(1-r)^2 + 4r\,|1-2x|}\right) \qquad r \in [0, 1] \tag{2}$$

Note that the parameter value $r = 1$ corresponds to a weakly intermittent situation.[5–9, 1] It turns out that the main features of the spectrum and the eigenfunctions depend only on the behavior of the map near the unstable fixed point 0 when r is close to 1, thus our results have a broader relevance, in fact, they bear in a sense a universal character.[3]

The paper is organized as follows. In Section 2 the intermittent situation ($r = 1$) is discussed, especially, contact is made with the quantum mechanical s-scattering on a Coulomb potential at zero energy. In Section 3 the nearly intermittent ($r \approx 1$) situation is considered. The problem is reduced approximately to the bound s-states in a Coulomb potential. Correction terms are calculated by standard quantum mechanical perturbation theory. Approaching intermittency, the number of the eigenstates essentially involved in the representation of a correlation function tends to the infinity, therefore the calculation of correlation functions is highly nontrivial, even if the spectral properties are already known.[3] In Section 4 we present and discuss an analytic expression for correlation functions, that is relevant near intermittency. The detailed derivation can be found in

Appendix B. A discussion of the results is given in the concluding Section 5. Appendix A contains an outline and discussion of the numerical method for calculating corrections in the intermittent situation.

2. THE INTERMITTENT SITUATION

One conclusion of our numerical study done on the family of maps (2) has been that finite size matrix approximations are poor near and at the intermittent situation, hence if one wants to get the spectral properties in those cases reliably, some other method is needed which does not involve any truncations of the matrix representations. As a first step we have chosen for an analytical study the intermittent map

$$x' = 1 - \sqrt{|1 - 2x|} \tag{3}$$

corresponding to $r = 1$ in Eq. (2). Using the basis $(1-x)^{4n+1}$ (where $n = 0, 1, 2, \ldots$); one obtains for the (j, k)-th matrix element of the Frobenius–Perron-operator (1) the simple closed expression

$$H_{j,k} = \binom{4k+1}{2j} 2^{-4k} \tag{4}$$

These matrix elements are displayed in Fig. 1. Our aim is to find an asymptotical solution to the eigenvalue equation, i.e., an expression of the eigenvector for large j-values, as the failure of the numerical calculation implies that this numerically inaccessible part of the eigenvector plays an inevitable role. The matrix representation (4) also supports this expectation as the largest matrix elements lie at the diagonal and decay with j as $j^{-1/2}$, while the strip of the non-negligible matrix elements along the diagonal has a width of $\approx \sqrt{j}$. Thus any truncation leads to a dramatic effect. Using Stirling's formula for the factorials arising in Eq. (4) for large j and $|k - j| \approx \sqrt{j}$ the $H_{j,k}$ matrix element is approximately given by

$$H_{j,k} = \sqrt{\frac{2}{\pi j}} \exp\left(-\frac{2m^2}{j}\right)\left(1 + \left(-\frac{3}{2}\frac{m}{j} + 2\frac{m^3}{j^2}\right) + \left(-\frac{5}{16}\frac{1}{j} + \frac{27}{8}\frac{m^2}{j^2} - \frac{16}{3}\frac{m^4}{j^3} + 2\frac{m^6}{j^4}\right)\right) \tag{5}$$

(where $m = k - j$), this expression being accurate up to the order $1/j$. The quality of the approximation is demonstrated in Fig. 2, where the difference

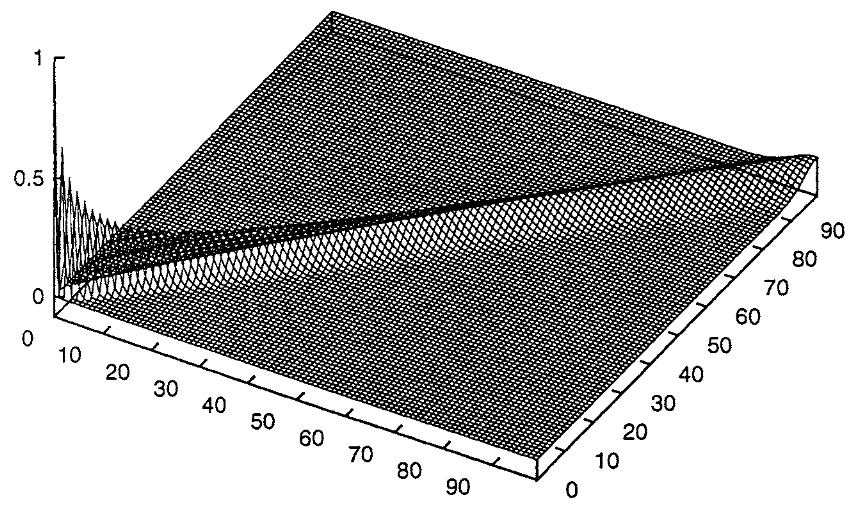

Fig. 1. The exact matrix elements $H_{j,k}$ in the intermittent case.

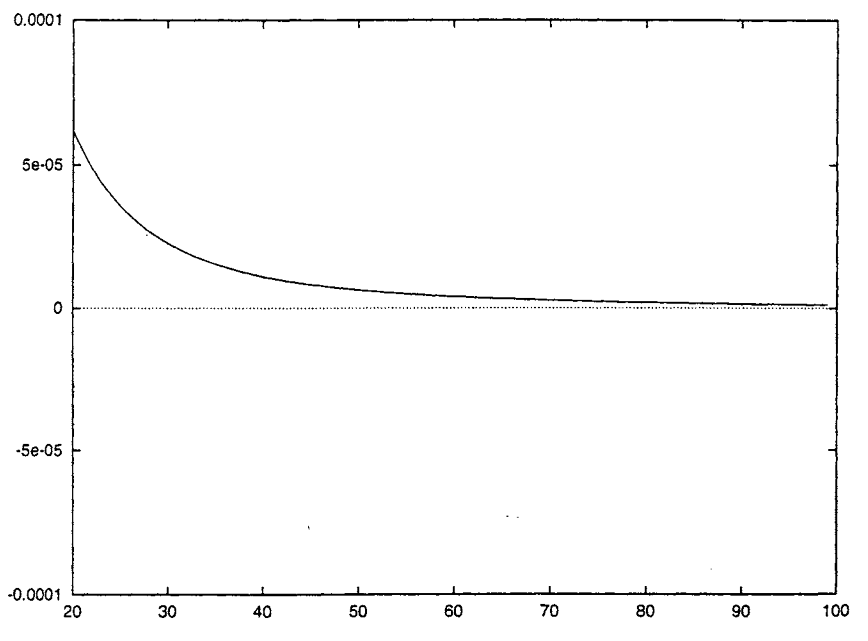

Fig. 2. The difference between the analytical approximation (5) and the exact values for the matrix elements $H_{j,k}$ in the intermittent case along the diagonal.

between the asymptotical expression (5) and the exact values (4) is shown. Retaining only the Gaussian, we find the approximate eigenvectors

$$a_k^{(1)} = \sin(C\sqrt{k}) \tag{6}$$

and

$$a_k^{(2)} = \cos(C\sqrt{k}) \tag{7}$$

both belonging to the eigenvalue

$$\lambda = \exp\left(-\frac{C^2}{32}\right) \tag{8}$$

This can be proven by inserting the above asymptotic expressions into the eigenvalue equation, replacing the summation over k with integration and evaluating it by using the saddle point method. The reason why we get this twofold degeneracy can be understood as a result of the asymptotical method which formally allows j and k to take on negative values as well, thus the situation is analogous with the continuous spectrum of a quantum particle performing unbounded one-dimensional motion. Actually these solutions get corrections near the origin (i.e., at $k = 0$), where the asymptotic expansion of $H_{j,k}$ does not hold, and, on the other hand they combine in a particular way to cancel each other for $k < 0$. As a result, the spectrum will not be degenerated, but is continuous. Using a quantum mechanical analogy again, the situation resembles to a one dimensional scattering process of a particle on a potential containing a hard core. It is interesting to note, that the asymptotic regime corresponds to the laminar motion in the original x-representation, as for large k the basis functions $(1-x)^{4k+1}$ are sharply peaked at the origin (which is now a marginally unstable fixed point) and nearly vanish elsewhere. The nonasymptotic regime, on the other hand, corresponds to the chaotic motion. Hence in this representations we are given a picture about intermittency that relates it to a scattering process, where the motion near the scattering potential corresponds to chaos. As we shall see later, in the nonintermittent situation one has to do with bound states in such a potential. This is in accordance with the fact that we have a discrete spectrum in that case.

Nevertheless, the above asymptotic solutions are not very precise numerically. To improve this approximation, one may seek for corrections proportional to $1/\sqrt{k}$ and to $1/k$. In order to get them, one has to take into account corrections to the saddle point method, which means that one uses Eq. (5) and retains terms like those proportional to m^2 which give a contribution when averaging them with the Gaussian. It turns out, however, that these corrections are proportional to $1/C$ and $1/C^2$, respectively, for $C \ll 1$, i.e., the correction terms diverge as the eigenvalue approaches 1 (which is the most interesting situation for us). On the other hand, the "improved" solution shows a remarkable feature, namely, that for small C it depends on C and on k only through the combination $C^2 k$:

$$a_k^{(i)} = h(C^2 k) \tag{9}$$

If one demands this dependence from the beginning and assumes that $C \ll 1$, $k \gg 1$ while no assumption is made about $C^2 k$, then one gets up to order $1/j$ in j and to order C^2 in C (defined now through Eq. (8)) the equation

$$h''(z) + \frac{1}{4z} h(z) = 0 \tag{10}$$

(z standing for $C^2 j$) whose two linearly independent solutions are (cf. ref. 10)

$$h^{(1)}(z) = \sqrt{z}\, J_1(\sqrt{z}) \tag{11}$$

and

$$h^{(2)}(z) = \sqrt{z}\, Y_1(\sqrt{z}) \tag{12}$$

J_1 and Y_1 standing for the first order Bessel and Neumann functions, respectively. Eq. (10) is identical with the radial Schrödinger equation for s-wave scattering in a Coulomb potential at zero energy. As discussed above, it is valid far from the "chaotic core" of the effective scattering potential which for small values of k gives rise to corrections. We shall see, however, that the range of validity of Eq. (10) extends down to $k = 1$ when $C \to 0$. The boundary conditions follow from the restrictions that the eigenfunctions in the original "x-space" should have an integrable singularity at $x = 0$ and their integral over the whole $[0, 1]$ interval should vanish. This latter follows from the orthogonality of the left and right eigenfunctions (belonging to different eigenvalues) and from the fact that the identically 1 function is the left eigenfunction of the Frobenius–Perron operator for $C = 0$ (i.e., for $\lambda = 1$). Note that it is valid only in the case of permanent chaos and does not apply for transient chaos. The eigenfunction of the Frobenius–Perron operator is given by

$$\phi_C(x) = \sum_{j=0}^{\infty} g_j (1-x)^{4j+1} \tag{13}$$

where the asymptotical form of g_j is given by some linear combination of $h^{(1)}$ and $h^{(2)}$ (see Eqs. (11), (12)). As the asymptotical form of both the Bessel and the Neumann function is a phase-shifted sinus with a one-over-square-root type amplitude, for fixed C and increasing j the moduli of the coefficients g_j grow like $j^{1/4}$. This growing is, however, superimposed by the exponential decay of the $(1-x)^{4j+1}$ factor, making the infinite sum in Eq. (13) absolutely convergent for $0<x\leqslant 1$. One can even show that the eigenfunction $\phi_C(x)$ is integrable near zero (one estimates the summation with an integral, after a term-by-term integration over x). This is true essentially because $\sum_{j=0}^{\infty} g_j/(4j+2)$ is finite. One can express the requirement of the integrability of the eigenfunction near zero as a boundary condition to Eq. (10) stating that $\int_1^{\infty} h(z)/z$ is finite (the lower limit of the integration range being an arbitrary positive value). Thus we can say that both propagating solutions (11) and (12) of Eq. (10) are allowed by the boundary condition at the infinity, as is usually the case in the customary scattering problems.

The next issue is how one can determine the actual eigenvector g_j for a given (small) C knowing the asymptotical solutions $h^{(i)}(C^2 j)$. The asymptotical solutions are actually rather accurate approximations even for small j-s when C is small. Nevertheless, at or near $j=0$ they do not satisfy the eigenvalue equation, thus giving rise to a correction term. If one starts with the proper linear combination of the two asymptotic solutions, then the correction term rapidly decays. This is the condition which selects the proper asymptotics. (If, however, not the proper linear combination has been chosen, then the correction term itself contains an asymptotic part, describing a "reflection" at the origin $j=0$.) The numerical procedure is outlined and discussed in Appendix A. According to those considerations, for small C, i.e., for an eigenvalue near unity the eigenfunction has the form

$$\phi_C(x)=2(1-x)-2\sum_{j=0}^{\infty} C\sqrt{j}\, J_1(C\sqrt{j})(1-x)^{4j+1} \tag{14}$$

(Here we have multiplied by -2 in order to exhibit the similarity to the $C=0$ case when just the first term of the r.h.s. of Eq. (14) remains.) The sum may be evaluated by replacing it with an integral. Neglecting terms proportional to C^2 times a nonsingular function (which come from the derivatives of the summand at $j=0$ when the Euler–Maclaurin Summation Formula is applied) one obtains (cf. ref. 11)

$$\phi_C(x)=2(1-x)-\frac{C^2(1-x)}{16\ln^2\left(\dfrac{1}{1-x}\right)}\exp\left(-\frac{C^2}{16\ln\left(\dfrac{1}{1-x}\right)}\right) \tag{15}$$

For $C \to 0$ the second term vanishes everywhere except near $x = 0$, where it has a sharp peak with unit area. As the first term is nothing but the normalized stationary probability density of the map (which is at the same time the eigenfunction of the Frobenius–Perron operator with unit eigenvalue (i.e., $C = 0$), one can see in which sense the limiting eigenfunction is approached and also, that the integral of the eigenfunction over x vanishes.

One can see now that the spectral properties around the upper edge of the spectrum depend predominantly on the laminar motion of the map near its marginally unstable fixed point, as the influence of the other parts of the map can show up itself through correction terms, which, however, become negligible when the eigenvalue approaches unity. On the other hand, corrections for larger values of C can be calculated with relatively little numerical efforts as the nondecaying asymptotical part of the eigenvectors (which has been previously the root of the numerical difficulties) is already taken into account analytically.

3. THE ASYMPTOTICAL SOLUTION OF THE FROBENIUS–PERRON EIGENVALUE EQUATION NEAR INTERMITTENCY

We seek for the asymptotical form of the matrix elements $H_{j,k}$ of the Frobenius–Perron operator $\hat{H}$ corresponding to thee family of maps (2) on the basis

$$\begin{aligned} \zeta_n(x) &= (f_l^{-1})'(x)(1 - 2f_l^{-1}(x))^{2n} \\ &= \left(\frac{r+1}{2} - rx\right)(1-x)^{2n}(1-rx)^{2n} \qquad (n = 0, 1, 2, \ldots) \end{aligned} \tag{16}$$

This form is suggested by the symmetry of the map as well as by the structure of the Frobenius—Perron operator (1). Indeed, for a symmetric fully developed chaotic map defined on the interval [0, 1] the Frobenius–Perron operator can be expressed as

$$(\hat{H}\varphi)(x) = (f_l^{-1})'(x)[\varphi(\tfrac{1}{2} - (\tfrac{1}{2} - f_l^{-1}(x))) + \varphi(\tfrac{1}{2} + (\tfrac{1}{2} - f_l^{-1}(x)))] \tag{17}$$

which implies that any polynomial $\varphi(x)$ goes over under the application of $\hat{H}$ into a linear combination of the functions $(f_l^{-1})'(x)(1 - 2f_l^{-1}(x))^{2n}$ which, if they are themselves polynomials, constitute a suitable basis. In our concrete example they coincide with those given above (Eq. (16)). One can see that this basis goes over for $r \to 1$ to that used in the intermittent situation.

Introducing the functions

$$h_k(x) = (\hat{H}\zeta_k)(x) \tag{18}$$

the matrix elements $H_{j,k}$ are defined by

$$h_k(x) = \sum_{j=0}^{\infty} H_{j,k} \zeta_j(x) \tag{19}$$

As one has to do here with polynomials, a numerical evaluation of the matrix elements for any given indices j, k is not difficult, however, unlike in the intermittent situation, a closed analytical expression (a counterpart of Eq. (4)) this time does not exist. Nonetheless an asymptotical expression can be derived. Our starting point now is Eq. (19), where we insert the expressions of the basis functions (16). Introducing the variable $z = 1 - 2f_l^{-1}(x) = (1-x)(1-rx)$ we get

$$\begin{aligned} &2^{-(4k+1)}(1+rz)(1+z)^{2k}(2-r+rz)^{2k} \\ &\quad + 2^{-(4k+1)}(1-rz)(1-z)^{2k}(2-r-rz)^{2k} = \sum_{j=0}^{\infty} H_{j,k} z^{2j} \end{aligned} \tag{20}$$

The exact matrix elements may be calculated by comparing the coefficients of the polynomials in both sides. The result is shown in Fig. 3.

In order to derive an asymptotical analytical approximation, we extend the variable z to the whole complex plane and determine the matrix elements by Cauchy's formula as

$$H_{j,k} = \frac{1}{2\pi i} \oint \tilde{h}_k(z)\, z^{-(2j+1)} \tag{21}$$

where the integration contour encircles the origin and $\tilde{h}_k(z)$ stands for the left hand side of Eq. (20). For large j and k values the integrals in Eq. (21) (each involving one of the terms of $\tilde{h}_k(z)$) may be evaluated by the saddle point method. It turns out that for a given j the matrix elements have a maximum versus k, namely

$$H_{j,k} \approx \frac{x}{\sqrt{\pi}} \sqrt{\frac{(1+r)^3}{2(1+2r-r^2)}} \exp\left(-\frac{(1+r)^3}{2(1+2r-r^2)} \left(\frac{k - \frac{2}{1+r} j}{\sqrt{j}} \right)^2 \right) \tag{22}$$

It has been assumed that $(k - (2/(1+r))j)/\sqrt{j}$ is of order unity (i.e., the expression (22) is valid near the maximum of the matrix elements in a strip of width $\approx \sqrt{j}$). The difference between the expression (22) and the exact matrix elements is displayed in Fig. 4. The most important difference between the intermittent and nonintermittent situations is that the maximal

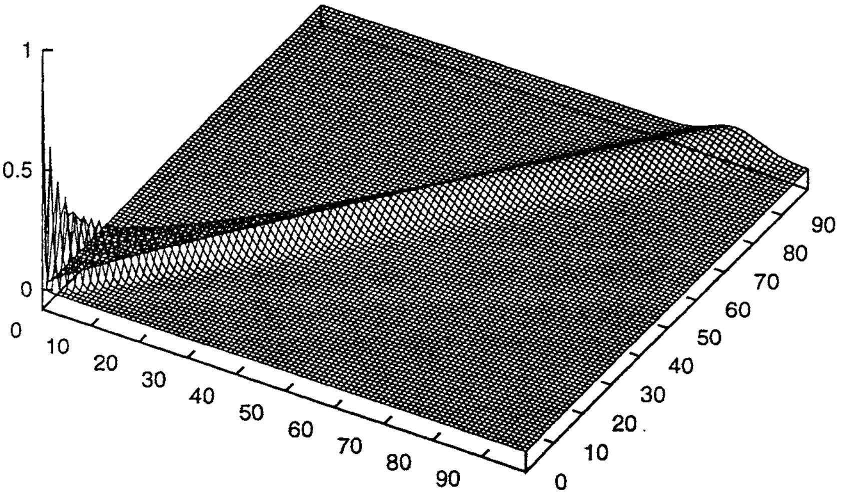

Fig. 3. The exact matrix elements $H_{j,k}$ in the nearly intermittent case ($r = 0.7$).

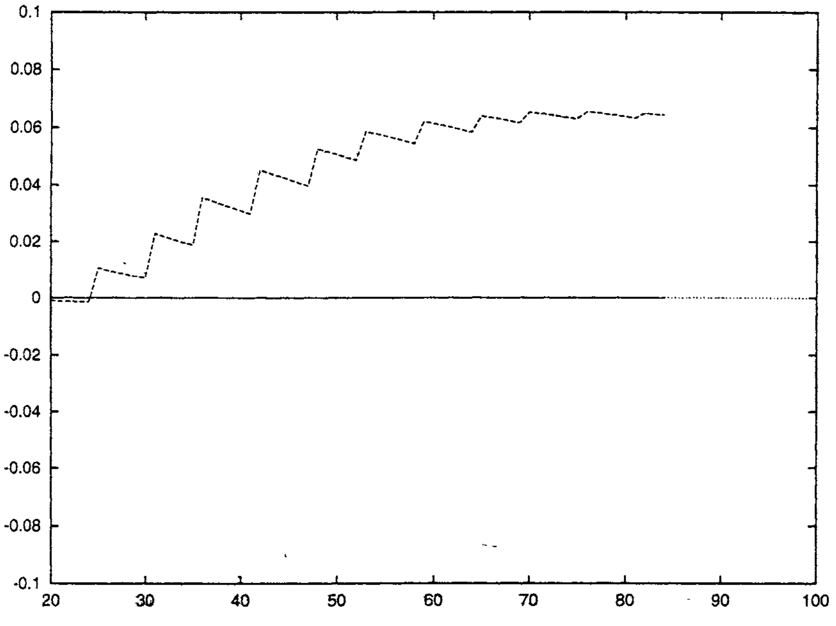

Fig. 4. Solid line: the difference between the analytical approximation (22) and the exact values for the matrix elements $H_{j,k}$ in the nearly intermittent case $r = 0.7$ along the maximum $k = (2/(1+r))j$. Dashed line: the same for the difference between the analytical approximation (26) and the exact matrix elements.

matrix elements lie now above the diagonal, and, when increasing the index j, their distance from the diagonal increases faster (linearly), than the spread of the significant matrix elements around the maximum (square root type increase). As a consequence, as we shall see, the eigenvectors a_k decay for large k exponentially. Therefore, we have to evaluate Eq. (21) not along $k \approx (2/(1+r))\,j$, as we would expect from Eq. (22), but along another line $k \approx bj$. The quantity b as well as the asymptotics $a_k \propto \exp(-\alpha k)$ of the eigenvector is to be determined from the condition

$$\sum_k H_{j,k} \exp(-\alpha k) \propto \exp(-\alpha j) \tag{23}$$

where the dominant terms of the sum come from the k values near bj. Explicitly, we get

$$b = \frac{2}{3-r} \tag{24}$$

and

$$\alpha = 2 \ln\left(\frac{2-r}{r}\right) \tag{25}$$

Evaluating the expression of the matrix element $H_{j,k}$ by the saddle point method around $k = bj$ in a strip of width $\sqrt{j}$, we get

$$\begin{aligned} H_{j,k} = {} & \frac{x}{\sqrt{\pi}} \sqrt{\frac{(3-r)^3}{2(1+2r-r^2)}} \exp\left(-2\frac{1-r}{3-r} \ln\left(\frac{2}{r}-1\right)\frac{1}{x^2}\right. \\ & \left. + 2p \ln\left(\frac{2}{r}-1\right)\frac{1}{x} - \frac{(3-r)^3 p^2}{2(1+2r-r^2)}\right) \\ & \times (1 + (b_{1,1}p + b_{1,3}p^3)x + (b_{2,0} + b_{2,2}p^2 + b_{2,4}p^4 + b_{2,6}p^6)x^2) \end{aligned} \tag{26}$$

where

$$b_{1,1} = \frac{(r^4 - 12r^3 + 36r^2 - 28r - 9)(3-r)}{4(1+2r-r^2)^2} \tag{27}$$

$$b_{1,3} = \frac{(r^4 - 16r^2 + 24r + 3)(3-r)^4}{12(1+2r-r^2)^3} \tag{28}$$

$$b_{2,0} = \frac{r^7 - 33r^6 + 235r^5 - 687r^4 + 835r^3 - 171r^2 - 243r - 57}{48(1+2r-r^2)^3} \tag{29}$$

$$b_{2,2} = -\frac{(3-r)^2}{32(1+2r-r^2)^4} \times (r^8 + 8r^7 - 240r^6 + 1496r^5 - 4010r^4 + 4664r^3 - 1216r^2 - 952r - 183) \tag{30}$$

$$b_{2,4} = -\frac{(4r^7 - 9r^6 - 122r^5 + 629r^4 - 1040r^3 + 459r^2 + 186r + 21)(3-r)^5}{24(1+2r-r^2)^5} \tag{31}$$

$$b_{2,6} = \frac{(3-r)^8 (r^4 - 16r^2 + 24r + 3)^2}{288(1+2r-r^2)^6} \tag{32}$$

and

$$x = \frac{1}{\sqrt{j}} \tag{33}$$

$$p = \frac{k - \frac{2}{3-r} j}{\sqrt{j}} \tag{34}$$

Fig. 5 shows the difference between the approximate expression (26) and the exact matrix elements.

Our next task is to solve the eigenvalue equation

$$\sum_{k=0}^{\infty} H_{j,k} s_k = \lambda s_j \tag{35}$$

We approximate the summation by an integral (considering k to be a continuous variable), extend its lower limit to $-\infty$, insert the asymptotical expression (22) for the matrix elements and evaluate the left hand side by the saddle point method. In order to make it systematically, we write

$$s_k = \varphi(k) \exp(-\alpha k) \tag{36}$$

and assume that $\varphi(k)$ grows (or decays) asymptotically at most like a power of k. Then we insert the expression (26) of $H_{j,k}$, expand $\varphi(k)$ around bj on the l.h.s. up to fourth order, and perform the integration. The result can be written down for arbitrary r, but for simplicity it will be presented for $r = 1 - \varepsilon$, where $\varepsilon \ll 1$, i.e., near the intermittent situation. As the l.h.s. of the equation becomes a function of bj, while at the r.h.s. $\varphi(j)$ stands, the latter should also be expanded around bj. Introducing the new independent variable

$$\tilde{x} = 4\varepsilon bj \tag{37}$$

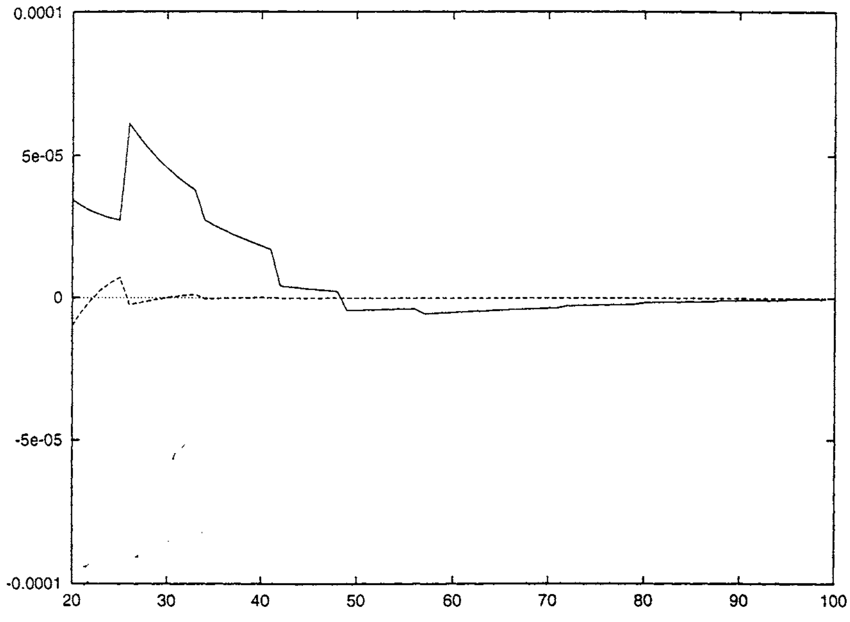

Fig. 5. Solid line: the difference between the analytical approximation (22) and the exact values for the matrix elements $H_{j,k}$ in the nearly intermittent case $r = 0.7$ along the line $k = (2/(3 - r))j$. Dashed line: the same for the difference between the analytical approximation (26) and the exact matrix elements.

and writing

$$\varphi(bj) = \xi(\tilde{x}) \tag{38}$$

we arrive at the equation

$$\tilde{x}\xi'' - \tilde{x}\xi' + \frac{2(1-\lambda)}{\varepsilon\lambda}\xi + \varepsilon\left[\frac{1}{4}\tilde{x}^2\xi'''' + \tilde{x}\xi''' + \left(-\frac{\tilde{x}^2}{4} - \tilde{x} + 2\right)\xi'' - 2\xi'\right] = 0 \tag{39}$$

For small ε the term proportional to ε may be neglected in the first approximation. The resulting equation,

$$\tilde{x}\xi'' - \tilde{x}\xi' + \frac{2(1-\lambda)}{\varepsilon\lambda}\xi = 0 \tag{40}$$

has a solution which diverges for $\tilde{x} \to \infty$ slower than exponentially (actually, as a power) only if $2(1-\lambda)/(\varepsilon\lambda)$ is a positive integer n. In this case

$$\xi = \xi_n = \tilde{x} L^1_{n-1}(\tilde{x}) \tag{41}$$

and the eigenvector $s_j^{(n)}$ (cf. Eqs. (36)–(38)) is expressed as

$$s_j^{(n)} = 4\varepsilon bj L^1_{n-1}(4\varepsilon b_j) \exp(-\alpha j) \tag{42}$$

where $L^1_{n-1}(\tilde{x})$ stands for the generalized Laguerre polynomial.[13] The eigenvalue λ is given (up to first order in ε) by

$$\lambda_n = 1 - \frac{\varepsilon}{2} n \tag{43}$$

Let us consider the transition to the intermittent case. Eq. (43) implies that in that case (i.e., when $\varepsilon \to 0$) the spacing between neighbouring eigenvalues vanishes, thus we get a continous spectrum. In order to get the eigenvectors, we fix the value of λ, thus also of εn and then take the limit $\varepsilon \to 0$, or, equivalently, $n \to \infty$. We obtain[12]

$$\begin{aligned} s_j^{\lambda} &= \lim_{n \to \infty} \frac{8(1-\lambda)\, bj}{n} L^{(1)}_{n-1} \left(\frac{8(1-\lambda)\, bj}{n} \right) e^{-\alpha j} \\ &= \sqrt{8(1-\lambda) j}\; J_1(2\sqrt{8(1-\lambda) j}) \end{aligned} \tag{44}$$

Provided that $1-\lambda \ll 1$, we may write according to Eq. (8) $1-\lambda \approx (C^2/32)$, thus we get

$$s_j^{\lambda} \approx \frac{C}{2} \sqrt{j}\; J_1(C\sqrt{j}) \tag{45}$$

in accordance with Eq. (11), which gives the dominant contribution near the upper edge of the spectrum (cf. the discussion after Eq. (12)).

When deriving Eqs. (41)–(43) we have made use of the same argument as that applied at the solution of the radial Schrödinger equation with a Coulomb potential. It is indeed possible to cast Eq. (40) to the same form. To do this, let us introduce the new independent variable

$$\rho = \nu \tilde{x} \tag{46}$$

and the new function

$$\chi(\rho) = \xi \exp\left(-\frac{\rho}{2\nu} \right) \tag{47}$$

where

$$\nu = \frac{2(1-\lambda)}{\varepsilon\lambda} \tag{48}$$

Then we get from Eq. (40)

$$-\chi'' - \frac{1}{\rho}\chi = -\frac{1}{4\nu^2}\chi \tag{49}$$

This is the well-known form of the radial Schrödinger equation. Note that the actual radial wave function $R(\rho)$ corresponds to χ/ρ. The term $-1/\rho$ represents an attracting Coulomb potential and $E = -1/(4\nu^2)$ is the Rydberg-formula. As usually, bound states are associated with positive integer values for ν. We may apply the transformation (46)–(47) also to Eq. (39). Before doing that we simplify somewhat the correction term. We shall be interested only in the next order correction to the eigenvalue (43), thus in the order ε term of Eq. (39) we may express all the higher derivatives in terms of ξ' and of ξ, as we may assume at the given accuracy that ξ satisfies Eq. (40). Thus we get for the n-th eigenvalue and eigenfunction

$$\tilde{x}\xi'' - \tilde{x}\xi' + \nu\xi + \varepsilon\left[\left(\frac{n(n+1)}{4} - \frac{3n}{2\tilde{x}}\right)\xi - \frac{n}{2}(1+\tilde{x})\,\xi'\right] = 0 \tag{50}$$

Performing now the transformation (46)–(47), we get

$$-\chi'' - \frac{1}{\rho}\chi - \varepsilon\left[\left(-\frac{3n}{2}\frac{1}{\rho^2} + \frac{n}{4}\frac{1}{\rho} - \frac{1}{4n}\right)\chi - \left(\frac{n}{2}\frac{1}{\rho} + \frac{1}{2}\right)\chi'\right] = -\frac{1}{4\nu^2}\chi \tag{51}$$

Applying first order perturbation theory, we may express the first correction of $-1/(4\nu^2)$ as the diagonal matrix element of the perturbing operator, i.e.,

$$\delta\left(-\frac{1}{4\nu^2}\right) = -\varepsilon\int_0^\infty d\rho\, \chi_n(\rho)\left[\left(-\frac{3n}{2}\frac{1}{\rho^2} + \frac{n}{4}\frac{1}{\rho} - \frac{1}{4n}\right)\chi_n(\rho) - \left(\frac{n}{2}\frac{1}{\rho} + \frac{1}{2}\right)\chi_n'(\rho)\right] \tag{52}$$

Using Eq. (48) this leads to the expression of the eigenvalue λ

$$\lambda = 1 - \frac{n}{2}\varepsilon - \left(\frac{7}{8}n - \frac{1}{8}n^2\right)\varepsilon^2 + O(\varepsilon^3) = \exp\left[-n\left(\frac{\varepsilon}{2} + \frac{7}{8}\varepsilon^2\right)\right] + O(\varepsilon^3) \tag{53}$$

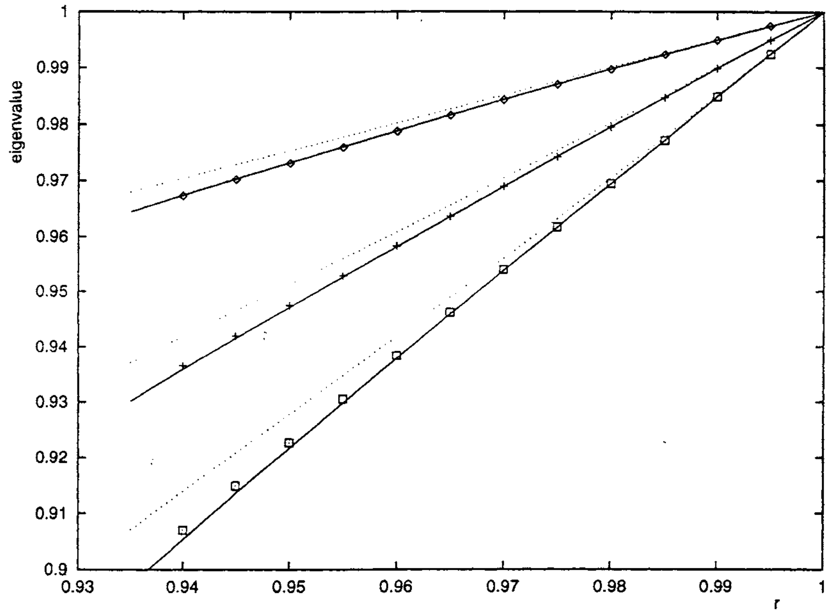

Fig. 6. Comparison of the analytical expression (53) with the results of a direct numerical measurement for the first three nontrivial eigenvalues near $r = 1$. Diamonds: λ_1, crosses: λ_2, squares: λ_3 (numerically measured data). The dotted and solid lines display the corresponding analytical results up to first and second order in ε, respectively.

which is valid in ε up to second order, provided that not only ε, but also εn is much smaller than 1. A comparison of these results with those of a direct numerical measurement[2] is displayed in Fig. 6.

As for a comparison in case of the eigenfunctions, first we have to determine them in coordinate space. Eqs. (42) and (16) imply

$$s^{(n)}(x) = \sum_{j=0}^{\infty} 4\varepsilon bj L_{n-1}^{(1)}(4\varepsilon bj) \exp(-\alpha j) \left(\frac{r+1}{2} - rx\right) (1-x)^{2j} (1-rx)^{2j}$$

$$\approx \frac{1}{4\varepsilon b} \left(\frac{r+1}{2} - rx\right) \int_0^{\infty} dy\, y L_{n-1}^{(1)}(y)\, e^{-\kappa(x) y} \tag{54}$$

where

$$\kappa(x) = \frac{1}{4\varepsilon b} [\alpha - 2 \ln((1-x)(1-rx))] \tag{55}$$

Using Rodrigues' formula[14]

$$L_n^{(1)}(x) = \frac{1}{n!} \frac{e^x}{x} \frac{d^n}{dx^n} (x^{n+1} e^{-x}) \tag{56}$$

we may evaluate the integral in (54) to get

$$s^{(n)}(x) = \frac{n}{4\varepsilon b} \left(\frac{r+1}{2} - rx \right) \frac{(\kappa(x)-1)^{n-1}}{\kappa(x)^{n+1}} \tag{57}$$

As our asymptotical method neglects any effects which appear at $j = 0$, the integral of these functions does not vanish. Therefore, a suitable multiple of $P(x) = 2\zeta_0(x)$ must be still subtracted. The resulting expression, however, gives a reasonable fit only quite near to the intermittent case $r = 1$, so that for $r < 0.99$ one should calculate corrections as well. The essence of the problem is not to push perturbation theory to higher order, but to take into account the second independent solution of the Eq. (39) and to add corrections near the origin, where the asymptotic expansion does not hold any longer. The procedure is in complete analogy with the intermittent situation (cf. Appendix A). It may be surprising that the second independent solution also plays a role. Indeed, at this point the analogy with the quantum mechanical case breaks down. The reason is that the physical meaning of the eigenfunctions is different, especially, they are differently related to probability distributions. Therefore in the quantum mechanical case a singularity in the origin is not allowed, while it is allowed in our case. Note that both solutions correspond to the same eigenvalue, i.e., the boundary condition at infinity completely determines the spectrum. The calculation of the second type of eigenfunctions is simpler after a Laplace transform, which brings us back to the original coordinate x. Indeed,

$$\begin{aligned} s^{(n)}(x) &= \sum_{j=0}^{\infty} \xi_n(4\varepsilon bj)\, e^{-\alpha j} \zeta_j(x) \\ &= \sum_{j=0}^{\infty} \xi_n(4\varepsilon bj)\, e^{-\alpha j} \left(\frac{r+1}{2} - rx \right) (1-x)^{2j} (1-rx)^{2j} \\ &\approx \left(\frac{r+1}{2} - rx \right) \frac{1}{4\varepsilon b} \int_0^\infty dz\, \xi_n(z)\, e^{-\kappa(x) z} \end{aligned} \tag{58}$$

where $\kappa(x)$ is given by Eq. (55). Taking the Laplace transform of Eq. (39) we get

$$\xi_n(0) + (n+1-2\kappa)\, \tilde{\xi}(\kappa) - \kappa(\kappa-1) \frac{d\tilde{\xi}(\kappa)}{d\kappa} = 0 \tag{59}$$

where

$$\tilde{\xi}(\kappa) = \int_0^\infty dz\, \xi_n(z)\, e^{-\kappa z} \tag{60}$$

stands for the Laplace transform of $\xi_n(z)$. Choosing $\xi_n(0) = 0$ in Eq. (59) and substituting (55) we arrive at the "regular" eigenfunctions (57) again. The "irregular" (logarithmic) eigenfunctions correspond to the choice $\xi_n(0) \neq 0$. Choosing $\xi_n(0) = -1$ we get

$$\tilde{\xi}(\kappa) = -\frac{(\kappa-1)^n}{\kappa^{n+1}} - n\ln(\kappa-1)\frac{(\kappa-1)^{n-1}}{\kappa^{n+1}} + \sum_{j=2}^{n} \frac{1}{j-1}\binom{n}{j}\frac{(\kappa-1)^{n-j}}{\kappa^{n+1}} \tag{61}$$

Taking the inverse Laplace transform we get the second independent solution for $\xi_n(z)$. It can be given as an infinite sum, but we do not reproduce here this cumbersome expression.

In order to get a reasonable approximation for the eigenfunctions even for relatively small r values, we apply the scheme decribed in Appendix A (cf. Eqs. (A1)–(A5)). Now the two independent asymptotical solutions

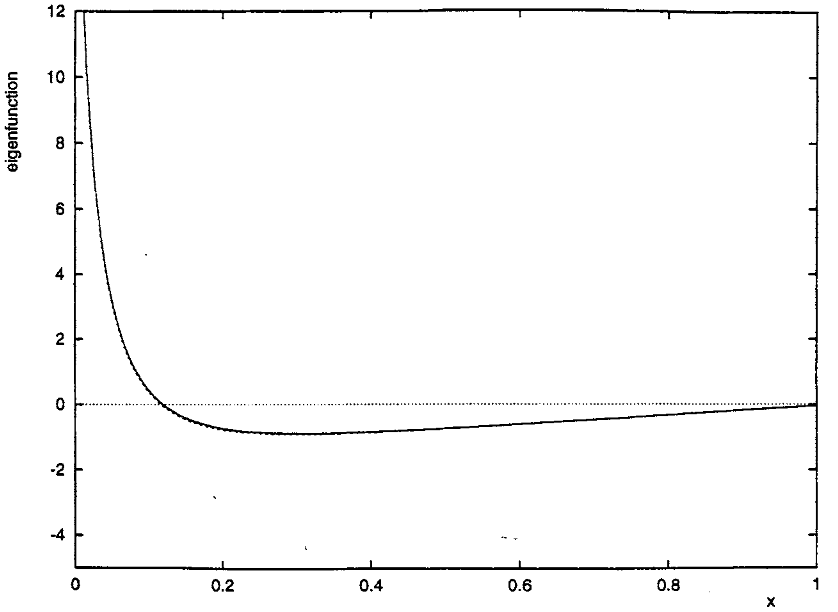

Fig. 7. Comparison of the analytical expression (57) with the results of a direct numerical measurement for the first nontrivial eigenfunction $s^{(1)}(x)$ at $r = 0.95$. Solid line: analytical approximation, dashed line: numerical result. The eigenfunction is normalized such that the integral of its modulus is unity.

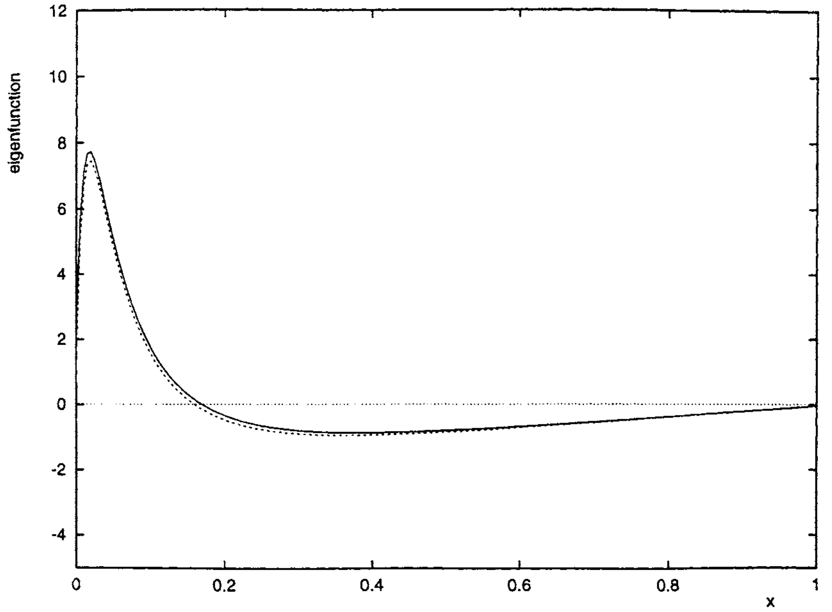

Fig. 8. Same for the second eigenfunction $s^{(2)}(x)$.

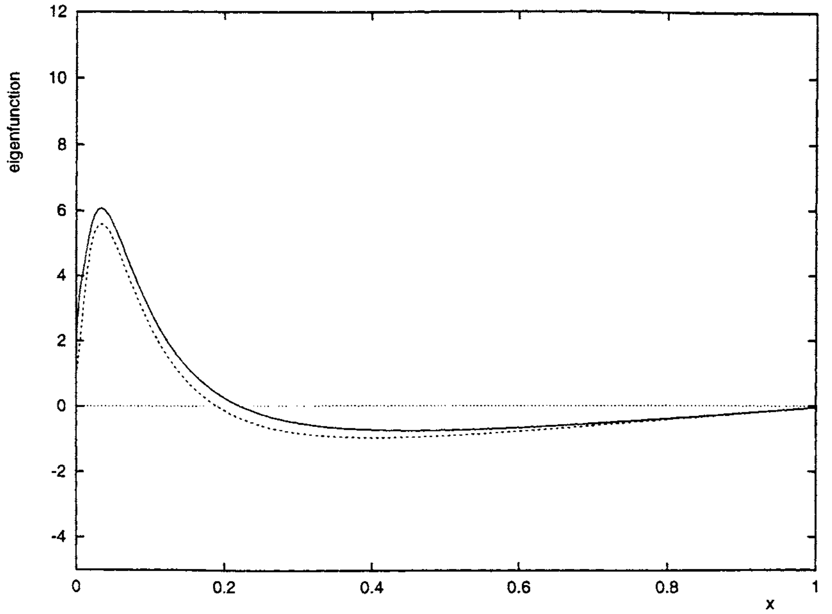

Fig. 9. Same for the third eigenfunction $s^{(3)}(x)$.

$h_j^{(1)}, h_j^{(2)}$ correspond to the regular and irregular solutions for $\xi_n(z)$, and the exact matrix elements $H_{j,k}$ valid in the nonintermittent case (cf. Eq. (21)) should be used. As a demonstration, in Figs. 7–9 the results for the first three eigenfunctions at $r = 0.95$ are compared with those of the numerical measurements. In this case the corrections played already an important role, 37 terms had to be included. We also found that it was essential to use good approximations for the eigenvalues (cf. Eq. (53)).

4. CORRELATION FUNCTIONS

Recalling the definition (1) of the Frobenius–Perron operator $\hat{H}$, one may cast a correlation function

$$C_t^{A,B} = \int_0^1 dx\, B(f^{[t]}(x))\, A(x)\, P(x) \tag{62}$$

($f^{[t]}(x)$ standing for the tth iterate of the mapping $f(x)$) to the form

$$C_t^{A,B} = \int_0^1 dx\, B(x)\, \hat{H}^t(A(x)\, P(x)) \tag{63}$$

Expanding now $A(x)\, P(x)$ in terms of the eigenfunctions of the Frobenius–Perron operator the action of $\hat{H}^t$ reduces to a multiplication of each term by the tth power of the corresponding eigenvalue. The details of the calculation outlined here are presented in Appendix B. There we make use of the previously introduced infinite matrix representation (19) and apply Eq. (42) for the eigenvectors. This is allowed under the assumption that we are close to an intermittent situation (i.e., corrections and the irregular Coulomb functions may be neglected) and also that the correlators $A(x)$ and $B(x)$ have already zero mean with respect to the natural measure, therefore, a substraction of a multiple of $P(x)$ from the eigenfunctions is not necessary. Then we apply identities for the generalized Laguerre polynomials, substitute sums with integrals (which is made possible again by the closeness of the intermittent situation), and finally end up with the expression

$$\begin{aligned} C_t^{A,B} \approx \int_0^1 dx\, P(x)\, B(x) & \frac{\tau(4\varepsilon b)^2}{[4\varepsilon b\tau + (1-\tau)(\alpha - 2\ln((1-x)(1-rx)))]^2} \\ & \times A\left(f\left(\frac{1}{2} - \frac{1}{2}\exp\left(-\frac{1}{2}\frac{1}{4\varepsilon b\tau + (1-\tau)(\alpha - 2\ln((1-x)(1-rx)))}\right.\right.\right. \\ & \left.\left.\left.\times [\alpha(1-\tau)(4\varepsilon b - \alpha) - 2\ln((1-x)(1-rx))(4\varepsilon b - \alpha(1-\tau))]\right)\right)\right) \end{aligned} \tag{64}$$

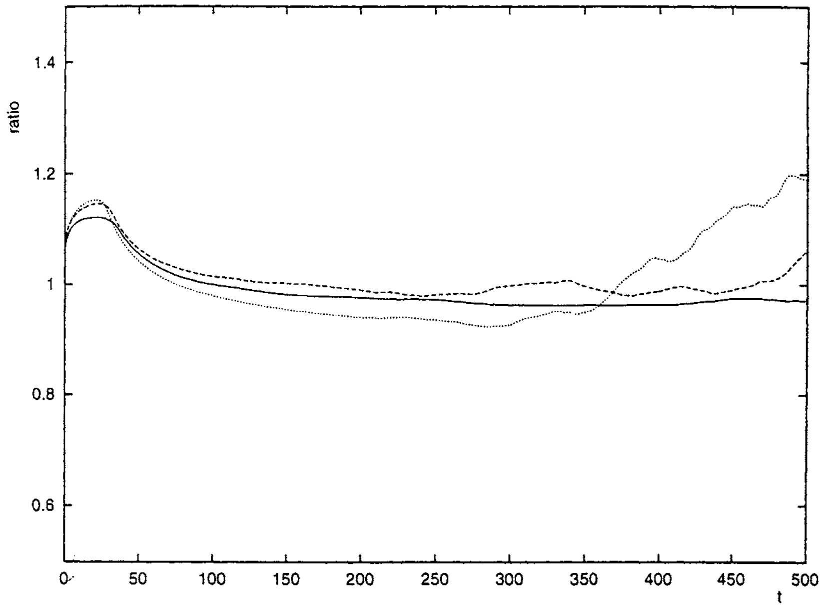

Fig. 10. The ratio between the numerically determined correlation function and the analytical expression (B24) at three control parameter values. Solid line: $r = 0.9999$, dashed line: $r = 0.99$, dotted line: $r = 0.98$. The functions $A(x)$ and $B(x)$ are given by Eq. (69).

Here

$$\tau = \exp\left(-\left(\frac{\varepsilon}{2} + \frac{7}{8}\varepsilon^2\right)t\right) \tag{65}$$

$$\varepsilon = 1 - r \tag{66}$$

$$\alpha = 2\ln\left(\frac{2-r}{r}\right) \tag{67}$$

$$b = \frac{2}{3-r} \tag{68}$$

A comparison with the results of a direct numerical measurement of the correlation function is displayed in Fig. 10 (cf. Fig. 1. in ref. 3, where the same is displayed with a rougher analytical estimate). The functions $A(x)$ and $B(x)$ are given by

$$A(x) = B(x) = \begin{cases} 1 & \text{if } x < 5 \times 10^{-2} \\ 0 & \text{otherwise} \end{cases} \tag{69}$$

5. CONCLUSION

A new analytical method has been developed for the determination of spectral properties near and at the intermittent situation of fully developed chaotic one-dimensional maps. We made use of an infinite dimensional matrix representation, determined the asymptotical expression of the matrix elements and reduced the eigenvalue equation to a differential equation, which in the lowest order coincides with the radial Schrödinger equation of the hydrogen atom for *s*-states. This allows for the calculation of corrections by using the standard quantum mechanical perturbation theory. We have demonstrated this by calculating the correction to the eigenvalue. The eigenfunctions were also analytically determined and compared with the results of numerical measurements. Finally, we derived an analytical expression for the correlation functions. It has been shown to fit well the numerical data.

One may think that our study applies only to a very specific example. Actually, the method may be applied to any fully developed chaotic one-dimensional map which is near to an intermittent situation and which has analytic inverse branches. The universal character of our results (within the class of mappings mentioned above) has been discussed in ref. 3. We expect that the method can be extended to the case of repellers, where one also has a complete symbolic dynamics. As a motivation we mention that in case of repellers a special interplay of transient chaos and intermittency appears, as discussed in the recent papers[18, 19] which the interested reader may consult.

APPENDIX A: CALCULATING CORRECTIONS IN THE INTERMITTENT SITUATION

The numerical procedure is the following: one assumes that for some n the correction term δg_j is negligible when $j > n$, and then determines both the proper asymptotics and δg_j from the requirement that the sum

$$\sum_{j=n/2+1}^{n} (\delta g_j)^2 \tag{A1}$$

be minimal (the elements in the second half of the correction vector are involved). Explicitely, it gives

$$\delta g_j = \cos(\alpha)\, \delta_j^{(1)} + \sin(\alpha)\, \delta_j^{(2)} \tag{A2}$$

and the proper asymptotics is given by

$$\cos(\alpha)\, h_j^{(1)} + \sin(\alpha)\, h_j^{(2)} \tag{A3}$$

the accurate eigenvector g_j being the difference of the above two quantities. Here $\delta_j^{(i)}$ is the solution of the equation

$$\sum_{k=0}^{n} H_{j,k}\, \delta_k^{(i)} - \lambda\, \delta_j^{(i)} = \sum_{k=0}^{\infty} H_{j,k} h^{(i)}(C^2 k) - \lambda h^{(i)}(C^2 j) \tag{A4}$$

These are the first n of the set of the eigenvalue equations. Note that if the correction vector δg_j is of length n (as is assumed here), its last $n/2$ components still enter the next n equations as well. Thus, when these components are small (as required), the error caused by them is small and the procedure is consistent. Indeed, for $C=0.1$ and $n=10$ these components are less than 10^{-5} (cf. Table I). The "phase shift" α is given by

$$\tan(2\alpha) = \frac{2 \sum_{j=n/2+1}^{n} \delta_j^{(1)} \delta_j^{(2)}}{\sum_{j=n/2+1}^{n} (\delta_j^{(1)})^2 - (\delta_j^{(2)})^2} \tag{A5}$$

as implied by Eqs. (A1), (A2). Numerically we get, as Table I demonstrates, that in case of $C \to 0$ the "phase shift" vanishes and $\delta g_j \to -\delta_{0,j}$. This result can be understood as follows. For $C \ll 1$ and for $0 < j \ll (1/C^2)$ the asymptotical solutions $h^{1,2}(z)$ can be written approximately as

$$\begin{aligned} h^{(1)}(z) &\approx \frac{1}{2} C^2 j \\ h^{(2)}(z) &\approx -\frac{2}{\pi} \end{aligned} \tag{A6}$$

hence the eigenvector g_j is of the form

$$g_j = \frac{1}{2} C^2 j \cos\alpha - \frac{2}{\pi} \sin\alpha + \delta g_j \tag{A7}$$

As the matrix elements $H_{j,k}$ are of order unity (compared to C), if $\alpha = O(C^2)$, then δg_j will be also of order C^2. For $j > (1/C^2)$ the asymptotical solutions are already accurate to order C^2 (provided that C is small enough), thus for those values of j δg_j practically vanishes. In view of Eq. (A4) this means that the above estimates for α and δg_j indeed hold. Note that for $j=0$ the combination $(1-\lambda)\, \delta g_0 \approx (C^2/32)\, \delta g_0$ enters the

Table I. Dependence of the Phase Shift and the Correlation Coefficients on the Parameter C

C	1.0	0.5	0.1	0.05
λ	0.96923323	0.99221794	0.99968755	0.99992188
α	−0.26334376	−0.04975108	−0.00103697	−0.00024860
δg_0	−0.55711388	−0.85478945	−0.99357489	−0.99831575
δg_1	0.09131282	0.01196515	0.00000560	−0.00000357
δg_2	0.05735793	0.00736996	−0.00001951	−0.00000816
δg_3	0.03704115	0.00553255	0.00003283	0.00000624
δg_4	0.01757775	0.00255459	−0.00000197	−0.00000159
δg_5	0.00679480	0.00111081	0.00000318	0.00000036
δg_6	−0.00025431	−0.00000901	−0.00000048	−0.00000012
δg_7	−0.00374950	−0.00063731	−0.00000183	−0.00000020
δg_8	−0.00472257	−0.00087973	−0.00000216	−0.00000018
δg_9	−0.00406673	−0.00084036	−0.00000206	−0.00000016

eigenvalue equation, hence (unlike when $j>0$) $\delta g_0 = O(1)$. Another important observation refers to the boundary conditions to Eq. (10): as we have seen, for $C \ll 1$ we get $h(z) \propto z$ (cf. Eqs. (11) and (A3)), just like in the case of the quantum mechanical scattering on a pure Coulomb potential.

APPENDIX B: DERIVATION OF AN ANALYTIC EXPRESSION FOR THE CORRELATION FUNCTIONS

$$C_t^{A,B} = \int_0^t dx\, B(x)\, \hat{H}^t(A(x)\, P(x))$$

$$= \int_0^1 dx\, B(x)\, \hat{H}^t\left(\sum_{j=0}^{\infty} a(j)\, \zeta_j(x)\right) \tag{B1}$$

$$= \int_0^1 dx\, B(x) \sum_{j=0}^{\infty} a(j) \sum_{k=0}^{\infty} (H^t)_{k,j}\, \zeta_k(x)$$

$$= \sum_{j,k} \left[\int_0^1 dx\, B(x)\, \zeta_k(x)\right] (H^t)_{k,j}\, a(j) \tag{B2}$$

Introducing

$$\beta(\tilde{x}) = \int_0^1 dx\, B(x)\, \zeta_{\tilde{x}/(4\varepsilon b)}(x) \tag{B3}$$

and

$$G_t(\tilde{x}, \tilde{y}) = \frac{1}{4\varepsilon b} (H^t)_{(\tilde{x}/(4\varepsilon b)),\, (\tilde{y}/(4\varepsilon b))} \tag{B4}$$

we may write Eq. (B1) as

$$C_t^{A,B} \approx \frac{1}{4\varepsilon b} \int_0^\infty d\tilde{x} \int_0^\infty d\tilde{y}\, \beta(\tilde{x})\, G_t(\tilde{x}, \tilde{y})\, a\left(\frac{\tilde{y}}{4\varepsilon b}\right) \tag{B5}$$

Making use of the eigenfunctions Eq. (42) we may write

$$\int_0^\infty d\tilde{y}\, G_t(\tilde{x}, \tilde{y})\, s^{(l)}_{\tilde{y}/(4\varepsilon b)} = \sum_j (H^t)_{k,j} s_j^{(l)} = \lambda_l^t s^{(l)}_{\tilde{x}/(4\varepsilon b)} \tag{B6}$$

Let us represent $a(\tilde{y}/(4\varepsilon b))$ (cf. Eq. (B5)) as

$$a\left(\frac{\tilde{y}}{4\varepsilon b}\right) = \sum_{l=1}^\infty c_l s^{(l)}_{\tilde{y}/(4\varepsilon b)} = \sum_{l=1}^\infty c_l \tilde{y} L^1_{l-1}(\tilde{y}) \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{y}\right) \tag{B7}$$

where

$$c_l = \frac{1}{l} \int_0^\infty d\tilde{y}\, a\left(\frac{\tilde{y}}{4\varepsilon b}\right) \exp\left(\frac{\alpha}{4\varepsilon b}\tilde{y}\right) \exp(-\tilde{y})\, L^1_{l-1}(\tilde{y}) \tag{B8}$$

Then we may write

$$\begin{aligned}
&\int_0^\infty d\tilde{y}\, G_t(\tilde{x}, \tilde{y})\, a\left(\frac{\tilde{y}}{4\varepsilon b}\right) \\
&\quad = \int_0^\infty d\tilde{y}\, G_t(\tilde{x}, \tilde{y}) \sum_{l=1}^\infty \frac{1}{l} \int_0^\infty d\tilde{y}' a\left(\frac{\tilde{y}'}{4\varepsilon b}\right) \\
&\qquad \times \exp\left(-\left(1 - \frac{\alpha}{4\varepsilon b}\right)\tilde{y}'\right) L^1_{l-1}(\tilde{y}')\, \tilde{y} L^1_{l-1}(\tilde{y}) \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{y}\right) \\
&\quad = \sum_{l=1}^\infty \frac{1}{l} \int_0^\infty d\tilde{y}'\, a\left(\frac{\tilde{y}'}{4\varepsilon b}\right) \\
&\qquad \times \exp\left(-\left(1 - \frac{\alpha}{4\varepsilon b}\right)\tilde{y}'\right) L^1_{l-1}(\tilde{y}')\, \lambda_l^t \tilde{x} L^1_{l-1}(\tilde{x}) \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{x}\right) \\
&\quad = \int_0^\infty d\tilde{y} \exp\left(-\left(1 - \frac{\alpha}{4\varepsilon b}\right)\tilde{y}\right) \tilde{x} \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{x}\right) \\
&\qquad \times \sum_{l=1}^\infty L^1_{l-1}(\tilde{x}) \frac{1}{l} L^1_{l-1}(\tilde{y})\, \lambda_l^t a\left(\frac{\tilde{y}}{4\varepsilon b}\right)
\end{aligned} \tag{B9}$$

that implies

$$G_t(\tilde{x}, \tilde{y}) = \tilde{x} \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{x} - \left(1 - \frac{\alpha}{4\varepsilon b}\right)\tilde{y}\right) \sum_{l=1}^{\infty} L^1_{l-1}(\tilde{x}) \frac{1}{l} L^1_{l-1}(\tilde{y})\, \lambda_l^t \quad \text{(B10)}$$

Here λ_l is of the form $\exp(-\gamma l)$ (cf. Eq. (53)) where $\gamma = \frac{1}{2}\varepsilon + \frac{7}{8}\varepsilon^2$. Inserting this and applying the identity[15]

$$\frac{1}{l} L^1_{l-1}(\tilde{y}) = \frac{1}{\tilde{y}} (L^0_{l-1}(\tilde{y}) - L^0_l(\tilde{y})) \quad \text{(B11)}$$

we obtain

$$G_t(\tilde{x}, \tilde{y}) = \frac{\tilde{x}}{\tilde{y}} \exp\left(-\frac{\alpha}{4\varepsilon b}\tilde{x} - \left(1 - \frac{\alpha}{4\varepsilon b}\right)\tilde{y}\right) \sum_{l=1}^{\infty} L^1_{l-1}(\tilde{x})(L^0_{l-1}(\tilde{y}) - L^0_l(\tilde{y}))\, \tau^l \quad \text{(B12)}$$

where τ stands for $\exp(-\gamma t)$. Let us insert the contour integral representation[16]

$$L^0_l(\tilde{y}) = \frac{\exp(\tilde{y})}{2\pi i} \oint_C dz\, \frac{\exp(-z)}{z - \tilde{y}} \left(\frac{z}{z - \tilde{y}}\right)^l \quad \text{(B13)}$$

where the contour C encircles the point $z = \tilde{y}$. We get

$$G_t(\tilde{x}, \tilde{y}) = -\frac{\tilde{x}\tau}{2\pi i} \exp\left(-\frac{\alpha}{4\varepsilon b}(\tilde{x} - \tilde{y})\right) \oint_C dz\, \frac{\exp(-z)}{(z - \tilde{y})^2} \sum_{l=1}^{\infty} L^1_{l-1}(\tilde{x}) \left(\frac{z\tau}{z - \tilde{y}}\right)^{l-1} \quad \text{(B14)}$$

Using the generating function of the Laguerre polynomials,[17] i.e., the identity

$$\sum_{l=1}^{\infty} L^1_{l-1}(\tilde{x})\, u^{l-1} = \frac{1}{(1-u)^2} \exp\left(\frac{\tilde{x}u}{u-1}\right) \quad \text{(B15)}$$

that holds true for

$$|u| < 1 \quad \text{(B16)}$$

we get finally

$$G_t(\tilde{x}, \tilde{y}) = -\frac{\tilde{x}}{\tilde{y}} \frac{\tau}{2\pi i} \exp\left(-\frac{\alpha}{4\varepsilon b}(\tilde{x} - \tilde{y})\right) \oint_{C'} dw\, \frac{\exp\left(-\tilde{y}w - \dfrac{\tilde{x}\tau w}{w(1-\tau) - 1}\right)}{(w(1-\tau) - 1)^2} \quad \text{(B17)}$$

Here the new complex variable $w = z/\tilde{y}$ has been introduced. The contour C' encircles accordingly the points $w = 1$ and $w = 1/(1-\tau)$, the latter coming from the condition (B16). Inserting Eqs. (B17), (B3), (16) into Eq. (B5) the integration over $\tilde{x}$ may be performed to get

$$
\begin{aligned}
&C_t^{A,B}\\
&\approx 2\varepsilon b\tau \int_0^1 dx\, P(x)\, B(x)\\
&\times\left(-\frac{1}{2\pi i}\right)\oint_{C'} dw\, \tilde{a}(4\varepsilon bw-\alpha)\\
&\times[w[4\varepsilon b\tau+(1-\tau)(\alpha-2\ln((1-x)(1-rx)))]-(\alpha-2\ln((1-x)(1-rx)))]^{-2}\\
&=-8\varepsilon^2 b^2\tau\int_0^1 dx\, P(x)\, B(x)\\
&\times\tilde{a}'\left(\frac{\alpha(1-\tau)(4\varepsilon b-\alpha)-2\ln((1-x)(1-rx))(4\varepsilon b-\alpha(1-\tau))}{4\varepsilon b\tau+(1-\tau)(\alpha-2\ln((1-x)(1-rx)))}\right)\\
&\times[4\varepsilon b\tau+(1-\tau)(\alpha-2\ln((1-x)(1-rx)))]^{-2}
\end{aligned}
\tag{B18}
$$

Here we have also used the fact that in our case $P(x) = 2(f_l^{-1})'(x)$. Furthermore,

$$\tilde{a}(p) = \int_0^\infty dy \exp(-yp)\frac{a(y)}{y} \tag{B19}$$

Consider now the expression

$$-\tilde{a}'(p) = \int_0^\infty dy \exp(-yp)\, a(y) \tag{B20}$$

$$\approx \sum_{j=0}^{\infty} a(j)\exp(-pj) \tag{B21}$$

Comparing this with the expansion (cf. (B1))

$$A(x)\, P(x) = \sum_{j=0}^{\infty} a(j)\,\zeta_j(x) = \sum_{j=0}^{\infty} a(j)\,\tfrac{1}{2}P(x)(1-2f_l^{-1}(x))^{2j} \tag{B22}$$

we get

$$-\tilde{a}'(p) \approx 2A\left(f\left(\frac{1}{2}\left(1-\exp\left(-\frac{p}{2}\right)\right)\right)\right) \tag{B23}$$

Inserting this into Eq. (B18) we get the final expression for the correlation function:

$$C_t^{A,B} \approx \int_0^1 dx\, P(x)\, B(x) \frac{\tau(4\varepsilon b)^2}{[4\varepsilon b\tau + (1-\tau)(\alpha - 2\ln((1-x)(1-rx)))]^2}$$
$$\times A\left(f\left(\frac{1}{2} - \frac{1}{2}\exp\left(-\frac{1}{2}\frac{1}{4\varepsilon b\tau + (1-\tau)(\alpha - 2\ln((1-x)(1-rx)))}\right.\right.\right.$$
$$\left.\left.\left.\times [\alpha(1-\tau)(4\varepsilon b - \alpha) - 2\ln((1-x)(1-rx))(4\varepsilon b - \alpha(1-\tau))]\right)\right)\right) \qquad \text{(B24)}$$

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